Understanding Orbits 4.1.3

In This Subsection You'll Learn to...

- Explain the basic concepts of orbital motion and describe how to analyze them
- Explain and use the basic laws of motion Isaac Newton developed
- Use Newton's laws of motion to develop a mathematical and geometric representation of orbits
- Use two constants of orbital motion—specific mechanical energy and specific angular momentum—to determine important orbital variables

Outline

Spacecraft work in orbits. We describe an orbit as a "racetrack" that a spacecraft drives around, as seen in Figure 4.1.3-1. Orbits and trajectories are two of the basic elements of any space mission. Understanding this motion may at first seem rather intimidating. After all, to fully describe orbital motion we need some basic physics along with a healthy dose of calculus and geometry. However, as we'll see, spacecraft orbits aren't all that different from the paths of baseballs pitched across home plate. In fact, in most cases, both can be described in terms of the single force pinning you to your chair right now—gravity.

Armed only with an understanding of this single pervasive force, we can predict, explain, and understand the motion of nearly all objects in space, from baseballs to spacecraft, to planets and even entire galaxies. Section 4.1.3 is just the beginning. Here we'll explore the basic tools for analyzing orbits. In the next several chapters we'll see that, in a way, understanding orbits gives us a crystal ball to see into the future. Once we know an object's position and velocity, as well as the nature of the local gravitational field, we can gaze into this crystal ball to predict where the object will be minutes, hours, or even years from now.

We'll begin by taking a conceptual approach to understanding orbits. Once we have a basic feel for how they work, we'll take a more rigorous approach to describing spacecraft motion. We'll use tools provided by Isaac Newton, who developed some fundamental laws more than 200 years ago that we can use to explain orbits today. Finally, we'll look at some interesting implications of orbital motion that allow us to describe their shape and determine which aspects remain constant when left undisturbed by outside non-gravitational forces.

Space Mission Architecture. This chapter deals with the Trajectories and Orbits segment of the Space Mission Architecture, introduced in Figure 1-20.

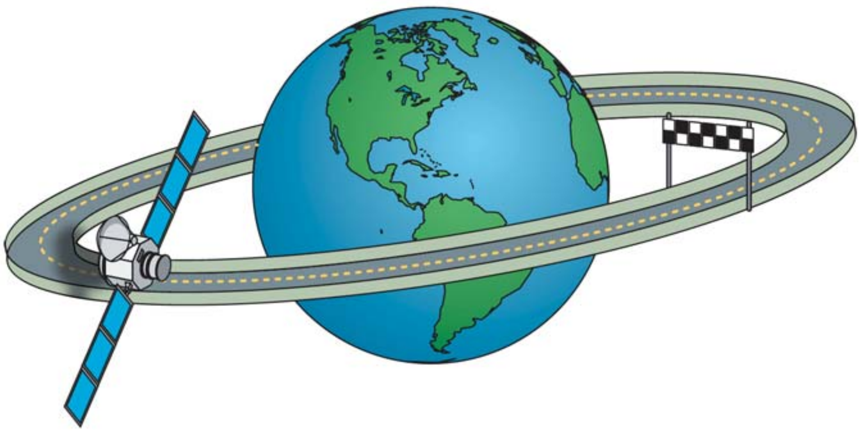


Figure 4.1.3-1. Orbits as Racetracks. Orbits are like giant racetracks on which spacecraft "drive" around Earth.

4.1.3.1 Orbital Motion

In This Section You'll Learn to...

- Explain, conceptually, how an object is put into orbit
- Describe how to analyze the motion of any object

Baseballs in Orbit

What is an orbit? Sure, we said it was a type of "racetrack" in space that an object drives around, but what makes these racetracks? Throughout the rest of this chapter we'll explore the physical principles that allow orbits to exist, as well as our mathematical representations of them. But before diving into a complicated explanation, let's begin with a simple experiment that illustrates, conceptually, how orbits work. To do this, imagine that we gather a bunch of baseballs and travel to the top of a tall mountain.

Visualize that we are standing on top of this mountain prepared to pitch baseballs to the east. As the balls sail off the summit, what do we see? Besides seeing unsuspecting hikers panting up the trail and running for cover, we should see that the balls follow a curved path. Why is this? The force of our throw is causing them to go outward, but the force of gravity is pulling them down. Therefore, the "compromise" shape of the baseball's path is a curve.

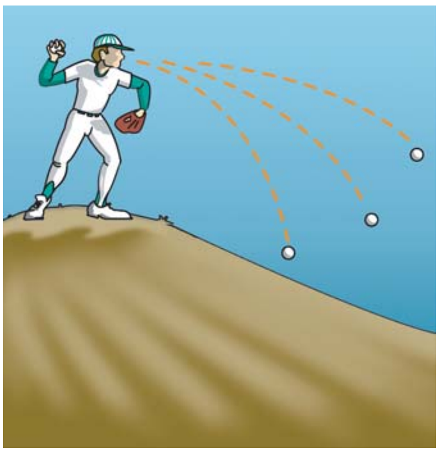


Figure 4.1.3-2. Throwing Baseballs Off of a Mountain. When we throw the balls faster, they travel farther before hitting the ground.

The faster we throw the balls, the farther they go before hitting the ground, as you can see in Figure 4.1.3-2. This could lead you to conclude that the faster we throw them the longer it takes before they hit the ground. But is this really the case? Let's try another experiment to see.

As we watch, two baseball players, standing on flat ground, will release baseballs. The first one simply drops a ball from a fixed height. At exactly the same time, the second player throws an identical ball horizontally at the same height as hard as possible. What will we see? If the second player throws a fast ball, it'll travel out about 20 m (60 ft.) or so before it hits the ground. But, the ball dropped by the first player will hit the ground at exactly the same time as the pitched ball, as Figure 4.1.3-3 shows!

How can this be? To understand this seeming paradox, we must recognize that, in this case, the motion in one direction is *independent* of motion in another. Thus, while the second player's ball is moving horizontally at 30 km/hr (20 m.p.h.) or so, it's still falling at the same rate as the first ball. This rate is the constant gravitational acceleration of all objects near Earth's surface, 9.798 m/s^2. Thus, they hit the ground at the same time. The only difference is that the pitched ball, because it also has horizontal velocity, will travel some horizontal distance before intercepting the ground.

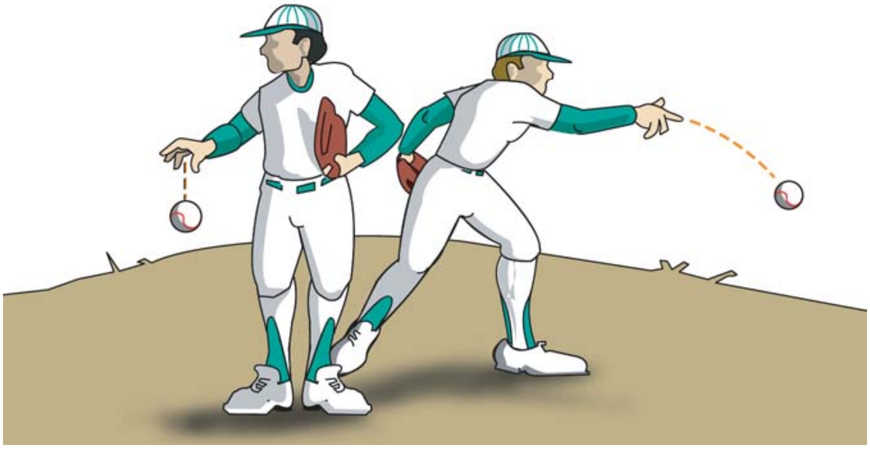


Figure 4.1.3-3. Both Balls Hit at the Same Time. A dropped ball and a ball thrown horizontally from the same height will hit the ground at the same time. This is because horizontal and vertical motion are independent. Gravity is acting on both balls equally, pulling them to the ground with exactly the same acceleration of 9.798 m/s^2.

Now let's return to the top of our mountain and start throwing our baseballs faster and faster to see what happens. No matter how fast we throw them, the balls still fall at the same rate. However, as we increase their horizontal velocity, they're able to travel farther and farther before they hit the ground. Because Earth is basically spherical in shape, something interesting happens. Earth's spherical shape causes the surface to drop approximately five meters vertically for every eight kilometers horizontally, as shown in Figure 4.1.3-4. So, if we were able to throw a baseball at 7.9 km/s (assuming no air resistance), its path would exactly match Earth's curvature. That is, gravity would pull it down about five meters for every eight kilometers it travels, and it would continue around Earth at a constant height. If we forget to duck, it may hit us in the back of the head about 85 minutes later. (Actually, because Earth rotates, it would miss us.) A ball thrown at a speed slower than 7.9 km/s falls faster than Earth curves away beneath it. Thus, it eventually hits the surface. The results of our baseball throwing experiment are shown in Figure 4.1.3-5.

If we analyze our various baseball trajectories, we see a whole range of different shapes. Only one velocity produces a perfectly circular trajectory. Slower velocities cause the trajectory to hit the Earth at some point. If we were to project this shape through the Earth, we'd find the trajectory is really a piece of an ellipse (it looks parabolic, but it's actually elliptical). Throwing a ball with a speed slightly faster than the circular velocity, also results in an ellipse. If we throw the ball too hard, it leaves Earth altogether on a parabolic or hyperbolic trajectory, never to return. No matter how hard we throw, our trajectory resembles either a circle, ellipse, parabola, or hyperbola. As we'll see in Section 4.1.3.4, these four shapes are *conic sections*.

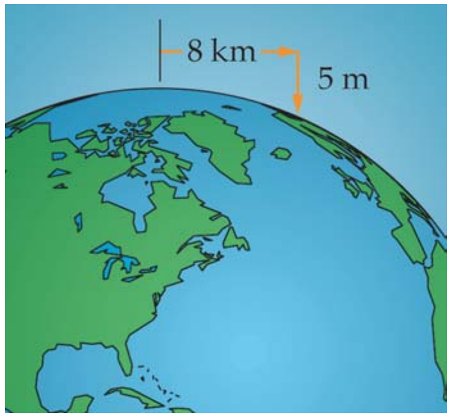


Figure 4.1.3-4. Earth's Curvature. Earth's curvature means the surface curves down about 5 m for every 8 km. On the surface of a sphere with that curvature, an object moving at 7.9 km/s is in orbit (ignoring air drag).

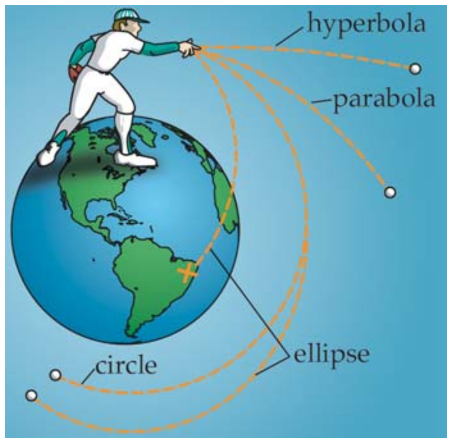


Figure 4.1.3-5. Baseballs in Orbit. As we throw baseballs faster and faster, eventually we can reach a speed at which Earth curves away as fast as the baseball falls, placing the ball in orbit. At exactly the right speed it will be in a circular orbit. A little faster and it's in an elliptical orbit. Even faster and it can escape Earth altogether on a parabolic or hyperbolic

So an object in orbit is literally falling around Earth, but because of its horizontal velocity it never quite hits the ground. Throughout this book we'll see how important having the right velocity at the right place is in determining the type of orbit we have.

Analyzing Motion

Now that we've looked at orbits conceptually, let's see how we can analyze this motion more rigorously. Chances are, when you first learned to play catch with a baseball, you had problems. Your poor partner had to chase after your first tentative throws, which never seemed to go where you wanted. But gradually, after a little bit of practice (and several exhausted partners), you got better. Eventually, you could place the ball right into your partner's glove, almost without conscious thought.

In fact, expert pitchers don't think about how to throw; they simply concentrate on where to throw. Somehow, their brain calculates the precise path needed to deliver the ball to the desired location. Then it commands the arm to a predetermined release point and time with exactly the right amount of force. All this happens in a matter of seconds, without a thought given to the likes of Isaac Newton and the equations that describe the baseball's motion. "So what?" you may wonder. Why bother with all the equations that describe *why* it travels the way it does?

Unfortunately, to build a pitching machine for a batting cage or to launch a spacecraft into orbit, we can't simply tell the machine or rocket to "take aim and throw." In the case of the rocket especially, we must carefully study its motion between the launch pad and space.

Now, we'll define a system for analyzing all types of motion. It's called the Motion Analysis Process (MAP) checklist and is shown in Figure 4.1.3-6. To put the MAP into action, imagine that you must describe the motion of a baseball thrown by our two baseball players in Figure 4.1.3-7. How will you go about it?

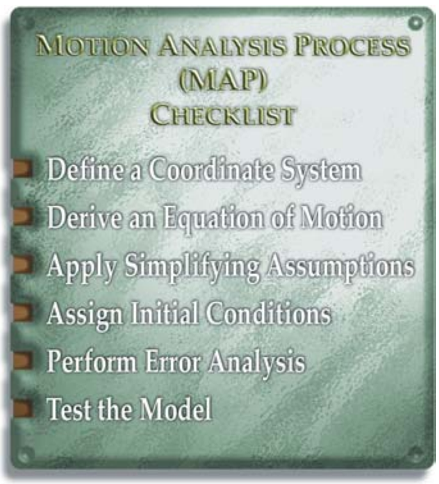


Figure 4.1.3-6. Motion Analysis Process (MAP) Checklist. Apply these steps to learn about moving objects and describe how they will move in the future.

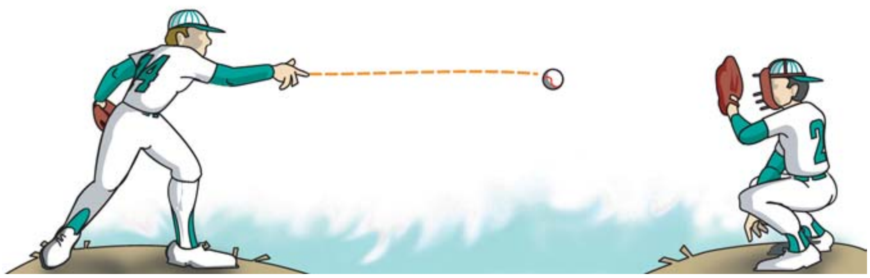


Figure 4.1.3-7. Baseball Motion. To analyze the motion of a baseball, or a spacecraft, we must step through the Motion Analysis Process (MAP) checklist.

First of all, you need to define some frame of reference or *coordinate system*. For example, do you want to describe the motion with respect to a nearby building or to the center of Earth? In either case, you must define a reference point and a coordinate frame for the motion you're describing, as shown in Figure 4.1.3-8.

Next you need some short-hand way of describing this motion and its relation to the forces involved—a short-hand way we'll call an *equation of motion*. Once you've determined what equation best describes the baseball's motion, you need to simplify it so you can use it. After all, you don't want to try to deal with how the motion of the baseball changes due to the gravitational pull of Venus or every little gust of wind in the park. So you must make some reasonable *simplifying assumptions*. For instance, you could easily assume that the gravitational attraction on the baseball from Venus, for example, is too small to worry about and the drag on the baseball due to air resistance is insignificant. And, in fact, as a good approximation, you could assume that the only force on the baseball comes from Earth's gravity.

With these assumptions made, you can then turn your attention to the finer details of the baseball problem. For example, you want to carefully define where and how the motion of the baseball begins. We call these the *initial conditions* of the problem. If you vary these initial conditions somehow (e.g., you throw the baseball a little harder or in a slightly different direction), the motion of the baseball will change. By assessing how these variations in initial conditions affect where the baseball goes, you can find out how sensitive the trajectory is to small changes or errors in them.

Finally, once you've completed all of these steps, you should verify the entire process by *testing the model* of baseball motion you've developed. Actually throw some baseballs, measure their trajectory deviations, and analyze differences (*error analysis*) between the motion you predict for the baseball and what you find from your tests. If you find significant differences, you may have to change your coordinate system, equation of motion, assumptions, initial conditions, or all of these. With the MAP in mind, we'll begin our investigation of orbital motion in the next section by considering some fundamental laws of motion Isaac Newton developed.

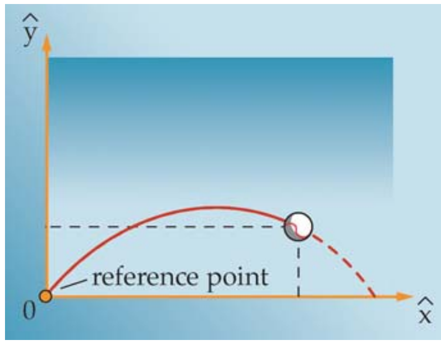


Figure 4.1.3-8. Defining a Coordinate System. To analyze a baseball's motion, we can define a simple, two-dimensional

Section Review

Key Concepts

- From a conceptual standpoint, orbital motion involves giving something enough horizontal velocity so that, by the time gravity pulls it down, it has traveled far enough to have Earth's surface curve away from it. As a result, it stays above the surface. An object in orbit is essentially falling around the Earth but going so fast it never hits it.
- The Motion Analysis Process is a general approach for understanding the motion of any object through space. It consists of
  - A coordinate system
  - An equation of motion
  - Simplifying assumptions
  - Initial conditions
  - Error analysis
  - Testing the model

4.1.3.2 Newton's Laws

In This Section You'll Learn to...

- Explain the concepts of weight, mass, and inertia
- Explain Newton's laws of motion
- Use Newton's laws to analyze the simple motion of objects

Figure 4.1.3-9. Cartwheel Galaxy. Our laws of motion apply universally, including the stars and planets of the Cartwheel Galaxy. *(Courtesy of the Association of Universities for Research in Astronomy, Inc./Space Telescope Science Institute)*

Since the first caveman threw a rock at a sabre-toothed tiger, we've been intrigued by the study of motion. In our quest to understand nature, we've looked for simple, fundamental laws that all objects obey. These Laws of Motion would apply universally for everything from gumdrops to galaxies (Figure 4.1.3-9). They would be unbreakable and empower us to explain the motion of the heavens, understand the paths of the stars, and predict the future position of our Earth. The Greek philosopher Aristotle defined concepts of orbital motion that held favor until challenged by such critical thinkers as Galileo and Kepler. Recall that Kepler gave us three laws to describe planetary motion, but didn't explain their causes. That's where Isaac Newton comes in.

Reflecting on his lifetime of scientific accomplishments, Newton rightly observed that he was able to do so much because he "stood on the shoulders of giants." Armed with Galileo's two basic principles of motion—inertia and relativity—and Kepler's laws of planetary motion, Isaac Newton was poised to determine the basic laws of motion that revolutionized our understanding of the world.

No single person has had as great an impact on science as Isaac Newton. His numerous discoveries and fundamental breakthroughs easily fill a volume the size of this book. Inventing calculus (math students still haven't forgiven him for that!), inventing the reflecting telescope, and defining gravity are just some of his many accomplishments. For our purposes, we'll see that the study of orbits (astrodynamics) builds on four of Newton's laws: three of motion and one describing gravity.

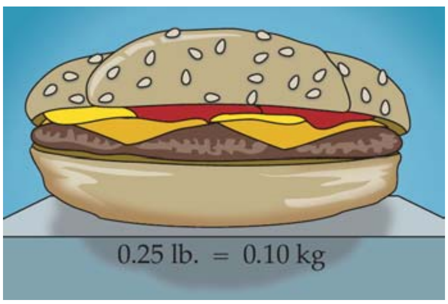


Figure 4.1.3-10. Quarter Pounder with Cheese™. When we order a Quarter Pounder with Cheese™, we get about 0.1 kg mass of meat.

Weight, Mass, and Inertia

Before plunging into a discussion of Newton's many laws, let's take a moment to complicate a topic that, until now, you probably thought you understood very well—*weight*. When we order a "Quarter Pounder with Cheese™" (Figure 4.1.3-10), we're describing the weight of the hamburger (before cooking). To measure this weight (say, to determine what it weighs after cooking), we slap the burger on a scale and read the results. If our scale gave weight in metric units, we'd see our quarter-pounder weighs about one newton. This property we call weight is really the result of another, more basic property of the hamburger called "mass" plus the influence of gravity. A hamburger that weighs one newton (1/4 pound) has a mass of 1/9.798 kg or about 0.1 kg. Knowing the mass of our hamburger, we automatically know three useful things about it, as illustrated in Figure 4.1.3-11.

First, *mass* is a measure of how much matter or "stuff" the hamburger contains. The more mass, the more stuff. If we have to haul 200 Quarter Pounders™ to a family picnic, we can add the masses of individual burgers to determine how much total mass we need to carry. Carrying these hamburgers, which have a total mass of 22.5 kg (50 lbs.), will take some planning. Thus, knowing how much stuff one object has is important whenever we must combine it with others (as we do for space missions).

But that's not all. Knowing the mass of an object also tells us how much inertia it has. Galileo first put forth the principle of *inertia* in terms of an object's tendency to stay at rest or in motion unless acted on by an outside influence. To visualize inertia, assume you're in "couch potato" mode in front of the TV, with your work sitting on the desk, calling for your attention. Somehow, you just can't motivate yourself to get up from the couch and start working. You have too much "inertia," so it takes an outside influence (another person or a deep-rooted fear of failure) to overcome that "inertia."

For a given quantity of mass, inertia works in much the same way. An object at rest has a certain amount of inertia, represented by its mass, that must be overcome to get it in motion. Thus, to get the Quarter Pounder™ from its package and into your mouth, you must overcome its inherent inertia. You do that when you pick it up, if you can!

An object already in motion also has inertia by virtue of its mass. To change its direction or speed, we must apply a force. For instance a car skidding on ice slides in a straight line indefinitely (assuming no friction force), or at least until it hits something.

Finally, knowing an object's mass reveals how it affects other objects merely by its presence. There's an old, corny riddle which asks "Which weighs more—a pound of feathers or a pound of lead?" Of course, they weigh the same—one pound. Why is that? Weight is a result of two things—the amount of mass, or "stuff," and gravity. So, assuming we measure the weight of feathers and lead at the same place, their masses are the same. *Gravity* is the tendency for two (or more) chunks of stuff to attract each other. The more stuff (or mass) they have, the more they attract. This natural attraction between chunks of stuff is always there. Thus, our Quarter Pounder™ lying in its package causes a very slight gravitational pull on our fries, milk shake, and all other mass in the universe. (You'd better eat fast!)

Now that you'll never be able to look at a Quarter Pounder™ the same way again, let's see how Isaac Newton used these concepts of mass to develop some basic laws of motion and gravity.

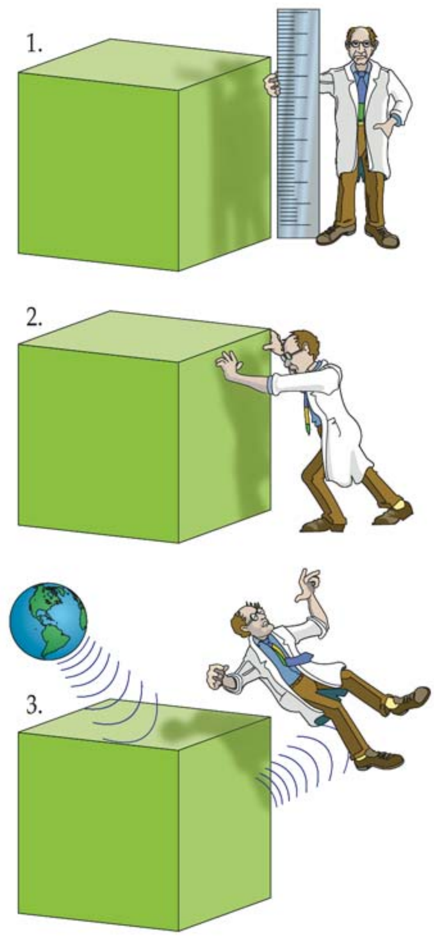


Figure 4.1.3-11. What is Mass? The amount of mass an object has tells us three things about it: (1) how much "stuff" it contains, (2) how much it resists changes in motion—its inertia, and (3) how much gravitational force it exerts and is exerted on it by other masses in the universe.

Momentum

Newton's First Law of Motion was actually a variation on Galileo's concept of inertia. He discovered it and other principles of gravity and motion in 1655, when a great plague ravaged England and caused universities to close. At the time, he was a 23-year-old student at Cambridge. Instead of hitting the beach for an extended "spring break," the more scholarly Newton hit the apple orchard for meditation (or so legend has it). But his findings weren't published until 1687—in *The Mathematical Principles of Natural Philosophy*. In this monumental work he stated

> ***Newton's First Law of Motion.*** *A body continues in its state of rest, or of uniform motion in a straight line, unless compelled to change that state by forces impressed upon it.*

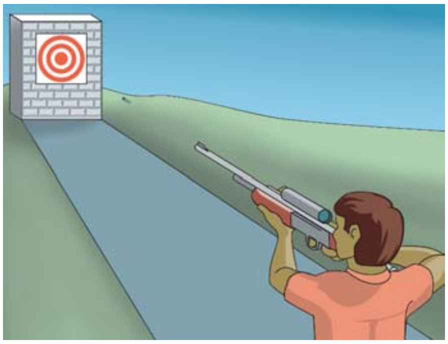


Figure 4.1.3-12. Newton's First Law. Any object in motion, such as a speeding bullet, will tend to stay in motion, in a straight line, unless acted on by some outside force (like gravity or hitting a brick wall.)

Newton's First Law says that any object (or chunk of mass) that is at rest will stay at rest forever, unless some force makes it move. Similarly, any object in motion will stay in motion forever, with a constant speed in the same straight-line direction, until some force makes it change either its speed or direction of motion. Try to stop a speeding bullet like the one in Figure 4.1.3-12 and you get a good idea how profound Newton's first law is.

One very important aspect of the first law to keep in mind, especially when you study spacecraft motion, is that motion tends to stay in a straight line. Therefore, if you ever see something not moving in a straight line, such as a spacecraft in orbit, some force must be acting on it.

We know that an object at rest is lazy; it doesn't want to start moving and will resist movement to the fullest extent of its mass. We've also discovered that, once it's in motion, it resists any change in its speed or direction. But the amount of resistance for an object at rest and one in motion are not the same! This seeming paradox is due to the concept of momentum. *Momentum* is the amount of resistance an object in motion has to changes in its speed or direction of motion. This momentum is the result of combining an object's mass with its velocity. Because an object's velocity can be either linear or angular, there are two types of momentum: linear and angular.

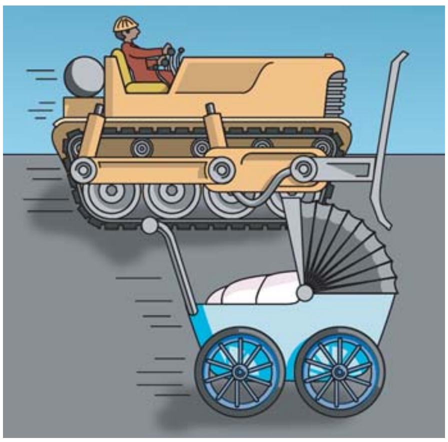


Figure 4.1.3-13. Bulldozer, Baby Carriage, and Momentum. The momentum of any object is the product of its mass and velocity. So, a bulldozer moving at the same speed as a baby carriage has much more momentum, due to its large mass.

Let's start with linear momentum. To see how it works, we consider the difference between a bulldozer and a baby carriage moving along a street, as shown in Figure 4.1.3-13. Bulldozers are massive machines designed to savagely rip tons of dirt from Earth. Baby carriages are delicate, four-wheeled carts designed to carry cute little babies around the neighborhood. Obviously, a bulldozer has much more mass than a baby carriage, but how does their momentum compare? Unlike inertia, which is a function only of an object's mass, *linear momentum,* $\vec{p}$, is the product of an object's mass, m, and its velocity, $\vec{V}$. [*Note:* because we describe velocity and momentum in terms of magnitude and direction, we treat them and other important concepts as *vector* quantities.]

$$\boxed{\vec{p} = m\vec{V}} \qquad (4.1.3\text{-}1)$$

where

$\vec{p}$ = linear momentum vector (kg · m/s)

m = mass (kg)

$\vec{V}$ = velocity vector (m/s)

To compare the linear momentum of the bulldozer and the baby carriage, we'd have to know how fast they were moving. For the two to have the same linear momentum, the baby carriage, being much less massive, would have to be going much, much faster!

Linear momentum is fairly basic because it involves motion in a straight line. Angular momentum, on the other hand, is slightly harder to understand because it deals with angular motion. Let's consider a simple toy top. If we set the top upright on a table, it will fall over, but if we spin it fast enough, the top will seem to defy gravity. A spinning object tends to resist changes in the direction and rate of spin, like the toy top shown in Figure 4.1.3-14, just as an object moving in a straight line resists changes to its speed and direction of motion. *Angular momentum, $\vec{H}$*, is the amount of resistance of a spinning object to change in spin rate or direction of spin. Linear momentum is the product of the object's mass, m, (which represents its inertia, or tendency to resist a change in speed and direction), and its velocity, $\vec{V}$. Similarly, angular momentum is the product of an object's resistance to change in spin rate or direction, and its rate of spin. An object's resistance to spin is its *moment of inertia, I*. We represent the *angular velocity*, which is a vector, by $\vec{\Omega}$. So we find the angular momentum vector, $\vec{H}$, using Equation (4.1.3-2).

$$\boxed{\vec{H} = I\,\vec{\Omega}} \qquad (4.1.3\text{-}2)$$

where

$\vec{H}$ = angular momentum vector (kg · m^2/s)

I = moment of inertia (kg · m^2)

$\vec{\Omega}$ = angular velocity vector (rad/s)

To characterize the direction of angular momentum, we need to examine the angular velocity, $\vec{\Omega}$. Look at the spinning wheel in Figure 4.1.3-15 and apply the right-hand rule. With our fingers curled in the direction it's spinning, our thumb points in the direction of the angular velocity vector, $\vec{\Omega}$, and the angular momentum vector, $\vec{H}$.

As Equation (4.1.3-2) implies, $\vec{H}$ is always in the same direction as the angular velocity vector, $\vec{\Omega}$. In the next section we'll see that, because of angular momentum, a spinning object resists change to its spin direction and spin rate.

We can describe angular momentum in another way. A mass spinning on the end of a string also has angular momentum. In this case, we find it by using the instantaneous tangential velocity of the spinning mass, $\vec{V}$, and the length of the string, $\vec{R}$, also called the *moment arm*. We combine these two with the mass, m, using a cross product relationship to get $\vec{H}$.

Note: We'll consistently refer to the velocity vector as $\vec{V}$, which denotes speed and direction. The magnitude of the velocity vector, we call speed or simply velocity, V.

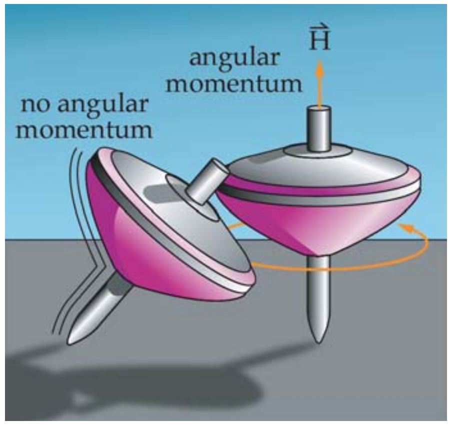


Figure 4.1.3-14. Angular Momentum. A non-spinning top (left) falls right over, but a spinning top, because of its angular momentum, resists the force applied by gravity

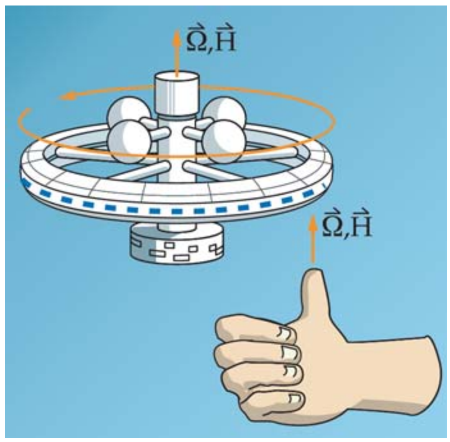


Figure 4.1.3-15. The Right-hand Rule. We find the direction of the angular velocity vector, $\vec{\Omega}$, and the angular momentum vector, $\vec{H}$, using the right-hand rule.

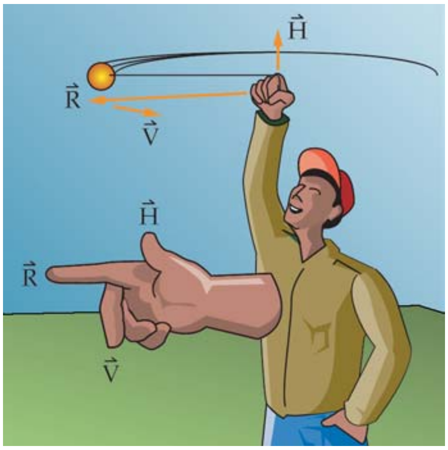


Figure 4.1.3-16. Describing Angular Momentum. The direction of the angular momentum vector, $\vec{H}$, is perpendicular to $\vec{R}$ and $\vec{V}$, and follows the right-hand rule.

$$\boxed{\vec{H} = \vec{R} \times m\vec{V}} \qquad (4.1.3\text{-}3)$$

where
$\vec{H}$ = angular momentum vector ($kg \cdot m^2/s$)
$\vec{R}$ = position (m)
m = mass (kg)
$\vec{V}$ = velocity vector (m/s)

By the nature of the cross product operation, we can tell that $\vec{H}$ must be perpendicular to both $\vec{R}$ and $\vec{V}$. Once again, we can use the right-hand rule to find $\vec{H}$, as shown in Figure 4.1.3-16.

In Section 4.1.3.5, we'll see that angular momentum is a very important property of spacecraft orbits. Later, we'll find angular momentum is also a useful property for gyroscopes and spacecraft in determining and maintaining their attitude.

Changing Momentum

Now that we've looked at momentum, let's go back to Newton's laws of motion. As we saw, whether we're dealing with linear or angular momentum, both represent the amount a moving object resists change in its direction or speed. Now we can determine what it will take to overcome this resistance using Newton's Second Law.

> ***Newton's Second Law of Motion.*** *The time rate of change of an object's momentum equals the applied force.*

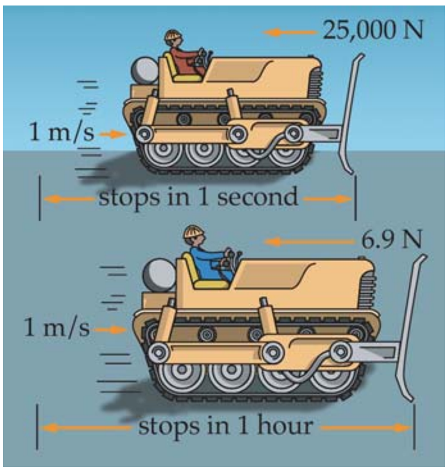


Figure 4.1.3-17. Newton's Second Law. The force we must apply to stop a moving object depends on how fast we want to change its momentum. If two bulldozers are moving at 1 m/s (about the speed of a brisk walk), we must apply a much, much larger force to stop a bulldozer in one second than to stop it in one hour.

In other words, to change an object's momentum very quickly, such as when we hit a fast ball with a bat, the force applied must be relatively high. On the other hand, if we're in no hurry to change the momentum, we can apply a much lower force over much more time.

Let's imagine we see a bulldozer creeping down the street at 1 m/s (3.28 ft./s), as in Figure 4.1.3-17. To stop the bulldozer dead in its tracks, we must apply some force, usually by pressing on the brakes. How much force depends on how fast we want to stop the bulldozer. If, for instance, we want to stop it in one second, we'd have to overcome all of its momentum quickly by applying a tremendous force. On the other hand, if we want to bring the bulldozer to a halt over one hour, we could apply a much smaller force. Thus, the larger the force applied to an object, the faster its momentum changes.

Now let's summarize the relationship implied by Newton's Second Law. The shorthand symbol we'll use to represent a force is $\vec{F}$. The symbol $\vec{p}$ represents linear momentum. To represent how fast a quantity is changing, we must introduce some notation from calculus. (See Appendix A.3 for a complete review of these concepts.) We use the Greek symbol "delta," Δ, to represent a very small change in any quantity. Thus, we represent the rate of change of a quantity, such as momentum, $\vec{p}$, over some short length of time, t, as

$$\frac{\Delta\vec{p}}{\Delta t} = \frac{\text{change in momentum}}{\text{change in time}} \tag{4.1.3-4}$$

This equation shows how fast momentum is changing. We now express Newton's Second Law in symbolic shorthand as

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\Delta(m\vec{V})}{\Delta t} \tag{4.1.3-5}$$

which is true only if Δt is very small.

We can expand this equation by applying the Δ to each term in the parentheses (another concept from calculus), to get

$$\vec{F} = m\frac{\Delta\vec{V}}{\Delta t} + \frac{\Delta m}{\Delta t}\vec{V} \tag{4.1.3-6}$$

So what can we do with this relationship? Let's begin with $\Delta m/\Delta t$ in the second term. This ratio represents how fast the mass of the object is changing. For many cases, the mass of the object won't change, so this term is zero for those cases. Now, for constant mass problems, we have only the first term in the relationship $\Delta\vec{V}/\Delta t$, which represents how fast velocity is changing. But this is just the definition of acceleration, $\vec{a}$. If we substitute $\vec{a}$ for $\Delta\vec{V}/\Delta t$ into Equation (4.1.3-6), we get the more familiar version

$$\boxed{\vec{F} = m\vec{a}} \tag{4.1.3-7}$$

where

$\vec{F}$ = force vector ($kg\ m/s^2 = N$)

m = mass (kg)

$\vec{a}$ = acceleration (m/s^2)

Equation (4.1.3-7) is arguably one of the most useful equations in all of physics and engineering. It allows us to understand how forces affect the motion of objects. Armed with this simple relationship, we can determine everything from how much force we need to stop a bulldozer, to the amount of acceleration Earth's gravity causes on the Moon.

Action and Reaction

Newton's first two laws alone would have made him famous, but he went on to discover a third law, which describes a very important relationship between action and reaction.

A simple example of Newton's Third Law in action applies to ice skating. Imagine two ice skaters, standing in the middle of the rink, as shown in Figure 4.1.3-18. If one gives the other a push, what happens? They both move backward! The first skater exerted a force on the second, but in turn an equal but opposite force is exerted on him, thus sending him backward! In fact, Newton found that the reaction is exactly equal in magnitude but opposite in direction to the original action.

Newton's Third Law of Motion. *When body A exerts a force on body B, body B will exert an equal, but opposite, force on body A.*

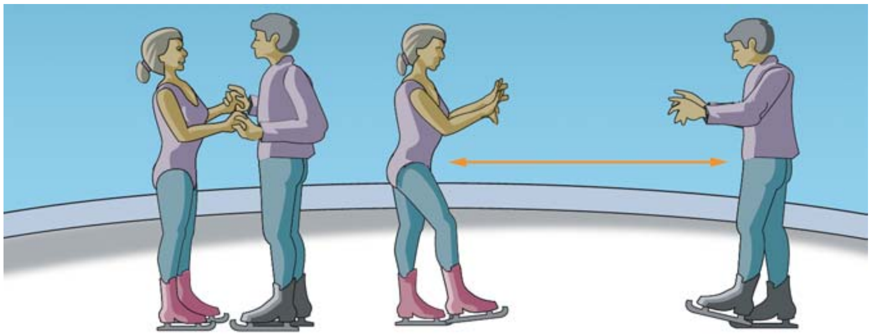


Figure 4.1.3-18. Two Ice Skaters Demonstrate Newton's Third Law. If they initially start at rest and the first one pushes against the second, they'll both go backward. The first skater applied an action—pushing—and received an equal but opposite reaction.

In the free-fall environment of space an astronaut must be very conscious of this fact. Suppose an astronaut tries to use a power wrench to turn a simple bolt without the force of gravity to anchor her in place. Unless she braces herself somehow, *she'll* start to spin instead of the bolt!

Gravity

The image most people have of Newton is of a curly-haired man clad in the tights and lace common to the 17th century, seated under an apple tree with an apple about to land on his head. After being hit by one too many apples, he suddenly jumped up and shouted "Eureka! (borrowing a phrase from Archimedes) I've invented gravity!" While this image is more the stuff of Hollywood than historical fact, it contains some truth. Newton did observe falling objects, such as apples, and read extensively Galileo's work on falling objects.

The breakthrough came when Newton reasoned that the force due to gravity must decrease with the square of the distance from the attracting body (Earth). In other words, an object twice as far away from Earth is attracted only one fourth as much. Newton excitedly took observations of the Moon to verify this model of gravity. Unfortunately, his measurements consistently disagreed with his model by one-sixth. Finally, in frustration, Newton abandoned his work on gravity. Years later, however, he found that the value for Earth's mass he had been using in his calculations was off by exactly one-sixth. Thus, his model of gravity had been correct all along! We call it Newton's Law of Universal Gravitation. "Universal" because we believe the same principle must apply everywhere in the universe. In fact, much of modern cosmology—all we know about the structure of the universe—depends on applying this simple law. We can see it applied most simply in Figure 4.1.3-19.

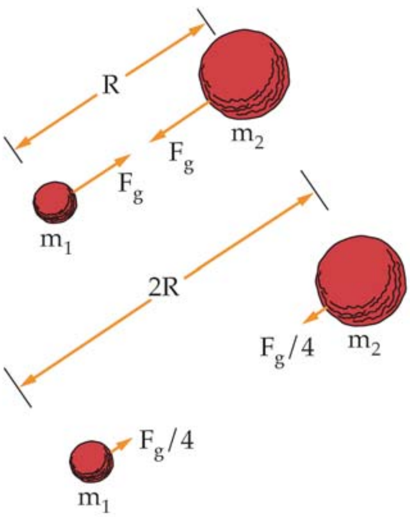


Figure 4.1.3-19. Newton's Law of Universal Gravitation. The force of attraction between any two masses is directly proportional to the product of their masses and inversely propor-tional to the square of the distance between them. Thus, if we double the distance between two objects, the gravitational force decreases to 1/4 the original amount.

Newton's Law of Universal Gravitation. *The force of gravity between two bodies is directly proportional to the product of their two masses and inversely proportional to the square of the distance between them.*

We can express this in symbolic shorthand as

$$\boxed{F_g = \frac{G\, m_1 m_2}{R^2}} \qquad (4.1.3\text{-}8)$$

where

F_g = force due to gravity (N)

G = universal gravitational constant = 6.67×10^{-11} N m^2/kg^2

m_1, m_2 = masses of two bodies (kg)

R = distance between the two bodies (m)

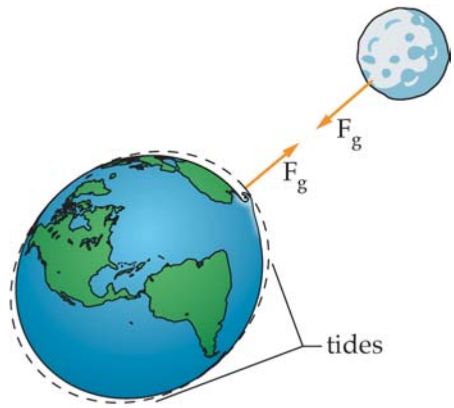


Figure 4.1.3-20. Earth and Moon in a Tug-of-War. Because of gravity, the Earth and Moon pull on each other with incredible force, which causes tides on Earth.

So what does this tell us? If we have two bodies, say Earth and the Moon, the force of attraction equals the product of their two masses, times a constant divided by the square of the distance between them. Let's look at some real numbers to see just how hard Earth tugs on the Moon and vice versa, as shown in Figure 4.1.3-20. Earth's mass, m_{Earth}, is 5.98×10^{24} kg (give or take a couple of mountains!), and the Moon's mass, m_{Moon}, is 7.35×10^{22} kg. The average distance between the Earth and Moon is about 3.84×10^8 m. We already know the gravitational constant, G. Using the relationship for gravitational force we just described, we find

$$F_g = \frac{G m_{Earth} m_{Moon}}{R^2}$$

$$F_g = \frac{\left(6.67 \times 10^{-11} \frac{Nm^2}{kg^2}\right)(5.98 \times 10^{24}\ kg)(7.35 \times 10^{22}\ kg)}{(3.84 \times 10^8\ m)^2}$$

$$F_g = 1.98 \times 10^{20}\ N \ \text{(or about } 4.46 \times 10^{19}\ lb_f)$$

In other words, there's a huge force pulling the Earth and Moon together. But do we experience the result of this age-old tug-of-war? You bet we do! The biggest result we see is in ocean tides. The side of Earth closest to the Moon is attracted more than the side away from the Moon (gravity decreases as the square of the distance). Thus, all the ocean water on the side closest to the Moon swells toward the Moon; on the other side, the water swells away from the Moon due to the conservation of angular momentum as Earth rotates. Depending on the height and shape of the ocean floor, tides can raise and lower the sea level in some places more than 5 m (16 ft.). If you think about how much force it would take you to lift half the ocean this much, the incredibly large force we computed above begins to make sense.

It's important to remember that the force of gravity decreases as the square of the distance between masses increases. This means that if you want to weigh less, you should take a trip to the mountains! If you normally live in Houston, Texas, (elevation ~0 ft.) and you take a trip to Leadville, Colorado, (elevation 3048 m or 10,000 ft.), you won't weigh as much. That's because you're a bit farther away from the attracting body (Earth's center). But before you start packing your bags, look closely at what is happening. Your *weight* will change because the force of gravity is slightly less, but your *mass* won't change. Remember, weight measures how much gravity is pulling you down. Mass measures how much stuff you have. So even though the force pulling down on the scale will be slightly less, you'll still have those unwanted bulges.

Because the gravitational force changes, the acceleration due to gravity also changes. We can compute the acceleration due to gravity by combining the relationships expressed in Newton's Second Law of Motion and Newton's Law of Universal Gravitation. We know from Newton's Second Law (dropping vector notation because we're interested only in magnitudes) that

$$F = ma \qquad (4.1.3\text{-}9)$$

We can substitute this expression into Newton's relationship for gravity (Equation (4.1.3-8)) to get an expression for the acceleration of any mass due to Earth's gravity.

$$m a_g = \frac{mG\, m_{Earth}}{R^2}$$

which simplifies to

$$a_g = \frac{G\, m_{Earth}}{R^2}$$

For convenience, we typically combine G and the mass of the central body (Earth in this case) to get a new value we call the *gravitational parameter, μ* (Greek, small mu), where $\mu \equiv G m$. For Earth, we denote this with a subscript, μ_{Earth}.

$$\boxed{a_g = \frac{\mu_{Earth}}{R^2}} \qquad (4.1.3\text{-}10)$$

where

a_g = acceleration due to gravity (m/s^2)

$\mu_{Earth} \equiv G\, m_{Earth} = 3.986 \times 10^{14}\ m^3/s^2$

R = distance to Earth's center (m)

If we substitute the values for μ_{Earth} and use Earth's mean radius (6,378,137.0 m) we get $a_g = 9.798\ m/s^2$ at Earth's surface, obviously pulling toward Earth's center. *Note:* we usually use kilometers instead of meters in this equation, because Earth's radius is so large.

Section Review

Key Concepts

- The mass of an object denotes three things about it
  - How much “stuff” it has
  - How much it resists motion—its inertia
  - How much gravitational attraction it has
- Newton’s three laws of motion are
  - **First Law.** A body continues in its state of rest, or in uniform motion in a straight line, unless compelled to change that state by forces impressed upon it.
    - The first law says that linear and angular momentum remain unchanged unless acted upon by an external force or torque, respectively
    - Linear momentum, $\vec{p}$, equals an object’s mass, m, times its velocity, $\vec{V}$
    - Angular momentum, $\vec{H}$, is the product of an object’s moment of inertia, I, (the amount it resists angular motion) and its angular velocity, $\vec{\boldsymbol{\Omega}}$
    - We express angular momentum as a vector cross product of an object’s position from the center of rotation, $\vec{R}$ (called its moment arm), and the product of its mass, m, and its instantaneous tangential velocity, $\vec{V}$
  - **Second Law.** The time rate of change of an object’s momentum equals the applied force.
  - **Third law.** When body A exerts a force on body B, body B exerts an equal but opposite force on body A.
- Newton’s Law of Universal Gravitation. The force of gravity between two bodies (m_1 and m_2) is directly proportional to the product of the two masses and inversely proportional to the square of the distance between them (R).
  - G = universal gravitational constant = $6.67 \times 10^{-11}\ Nm^2/kg^2$
  - We often use the gravitational parameter, μ, to replace G and m. $\mu \equiv G\,m$
    - The gravitational parameter of Earth, μ_{Earth}, is

      $\mu_{Earth} \equiv G\ m_{Earth} = 3.986 \times 10^{14}\ m^3/s^2$, or, using kilometer instead of meters, $\mu_{Earth} = 3.986 \times 10^5\ km^3/s^2$

4.1.3.3 Laws of Conservation

In This Section You'll Learn to...

- ✔ Describe the basic laws of conservation of momentum and energy and apply them to simple problems

For any mechanical system, basic properties, such as momentum and energy, remain constant. In physics we say that if a certain property or quantity remains unchanged for a given system, that property or quantity is *conserved*. So let's take a look at two basic properties—momentum and energy—to see how they're conserved.

Momentum

One very important implication of Newton's Third Law has to do with the amount of momentum in a system. Newton's Third Law implies the total momentum in a system remains unchanged, or is conserved. We call this *conservation of momentum*.

To understand this concept let's go back to our ice skating example. When the two skaters faced each other, neither of them was moving, so the total momentum of the system was zero. Then the first one pushed on the second, and he moved in one direction with some speed, while she moved in the other. Their speeds won't be the same unless their masses are equal. The first skater moves in one direction with a speed that depends on his mass, while the other moves in the opposite direction with a speed depending on her mass. Now, the second skater's momentum (the product of her mass and velocity) is equal in magnitude, but opposite in direction, to his.

Depending on how we define our frame of reference, the first skater's momentum could be negative while the other's is positive. Adding the momentums, gives us zero, so, the original momentum of the system (the two skaters) hasn't changed. Thus, as Figure 4.1.3-21 shows, we say that the system's total momentum is conserved. Example 4-4 also shows this principle in action.

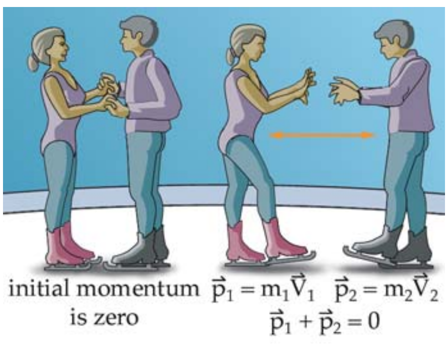


Figure 4.1.3-21. Conservation of Momentum. Two people on ice skates demonstrate the concept of conservation of linear momentum. Initially the two are at rest; thus, the momentum of the system is zero. But as one skater pushes on the other, they both start moving in opposite directions. Adding their two momentum vectors together still gives us zero; thus, momentum of the system is conserved.

This conservation principle works equally well for angular momentum. You've probably seen a good example of this with figure skaters, who always include a spin in their routines. Remember, once an object (or skater) begins to spin, it has angular momentum.

By watching these skaters closely, you may see them move their arms outward or inward to vary their spin rate. How does this change their spin rate? We know from Equation (4.1.3-2) that angular momentum, $\vec{H}$, equals the product of the moment of inertia, I, and the spin rate, $\vec{\Omega}$. The moment of inertia of an object is proportional to its distance from the axis of rotation. To change their moment of inertia, skaters move their arms outward or inward, which increases or decreases the radius, thereby changing I. Because momentum is conserved, it must stay constant as moment of inertia changes. But the only way this can happen is for the angular velocity, $\vec{\Omega}$, to change. Thus, if skaters put their arms out, as in Figure 4.1.3-22, they increase their moment of inertia and spin slower to maintain the same angular momentum. If they bring their arms in, as in Figure 4.1.3-23, they decrease their moment of inertia and increase the spin rate to maintain the same angular momentum.

Figure 4.1.3-22. Spinning Slowly. Skaters extend their arms to increase moment of inertia—spinning more slowly.

Figure 4.1.3-23. Spinning Quickly. Skaters bring in their arms to decrease moment of inertia—spinning faster. Total angular momentum is the same in both cases.

Energy

We've all had those days when somehow we just don't seem to have any energy. But what exactly is energy? Energy can take many forms including electrical, chemical, nuclear, and mechanical. For now, let's deal only with mechanical energy because it's the most important for understanding motion. If you've jumped off a platform, climbed a ladder, or played with a spring, you've experienced mechanical energy. *Total mechanical energy, E*, comes from an object's position and motion. It's composed of *potential energy, PE*, which is due entirely to an object's position and *kinetic energy, KE*, which is due entirely to the object's motion. Total mechanical energy can be only potential, kinetic, or some combination of both

$$\boxed{E = KE + PE} \tag{4.1.3-11}$$

where

E = total mechanical energy (kg m^2/s^2)
PE = potential energy (kg m^2/s^2)
KE = kinetic energy (kg m^2/s^2)

To better understand what the trade-off between potential and kinetic energy means, we need to understand where it takes place. We say that gravity is a *conservative field*—a field in which total energy is *conserved*. Thus, the sum of PE and KE, or the total E, in a conservative field is constant.

Potential energy is the energy an object in a conservative field has entirely because of its position. We call it "potential" energy because we don't really notice it until something changes. For example, if you pick up a 1 kg (2.2 lb.) mass and raise it above your head, it's higher position gives it more "potential" energy. This potential is realized when you drop the mass and it lands on your foot! To quantify this form of energy, we must derive an expression for the amount of work done by raising the object above a reference point (usually Earth's surface) against the force of gravity. If we raise the object a small distance (a few hundred meters or less), we can assume gravity is constant and we get

$$PE = m\, a_g h \qquad (4.1.3\text{-}12)$$

where

m = mass (kg)

a_g = acceleration due to gravity (m/s^2)

h = height above a reference point (m)

Thus, to compute an object's potential energy after raising it a small distance, we need to know three things: the amount of mass, m; its position above a reference point, h; and the acceleration due to gravity, a_g, at that reference point. But, if we want to find a spacecraft's potential energy in orbit high above Earth, we can't assume gravity is constant, and we can't use Earth's surface as a convenient reference point anymore. Let's see how we find potential energy in an orbit.

As we know from the last section, the gravitational acceleration varies depending on an object's distance from Earth's center, R. To derive the potential energy equation for this gravitational field, we must determine the amount of work it would take to move the spacecraft from Earth's center to its orbital position, a distance of R. That derivation yields

$$\boxed{PE = -\frac{m\mu}{R}} \qquad (4.1.3\text{-}13)$$

where

PE = spacecraft's potential energy ($kg\ km^2/s^2$)

m = spacecraft's mass (kg)

μ = gravitational parameter (km^3/s^2) = $3.986 \times 10^5\ km^3/s^2$

R = spacecraft's distance from Earth's center (km)

Notice the negative sign in Equation (4.1.3-13). This sign is due to the convention we're using, which defines R to be positive outward from Earth's center. We know potential energy should increase as we raise a spacecraft to a higher orbit, so is this still consistent? Yes! As we raise our spacecraft's orbit, R gets bigger, and PE gets less negative—which means it gets bigger too. Remember, for potential energy, –3 is a bigger quantity than –4 because it's less negative. (This approach is analogous to heat: an ice cube at –3 degrees Celsius is "hotter" than one at –4 degrees Celsius.) At the extreme, when R reaches infinity (or close enough), PE approaches zero.

One way to visualize this strange situation is to think about Earth's center being at the bottom of a deep, deep well (Figure 4.1.3-24). At the bottom of the well, R is zero, so PE is at a minimum (its largest negative value, PE = –∞). As we begin to climb out of the well, our PE begins to increase (gets less negative) until we reach the lip of the well at R near infinity. At this point, our PE is effectively zero, and for all practical purposes, we have left Earth's gravitational influence. Of course, we never really reach an "infinite" distance from Earth, but as we'll see when we discuss interplanetary travel, we essentially leave Earth's "gravity well" at a distance of about one million km (621,400 mi.).

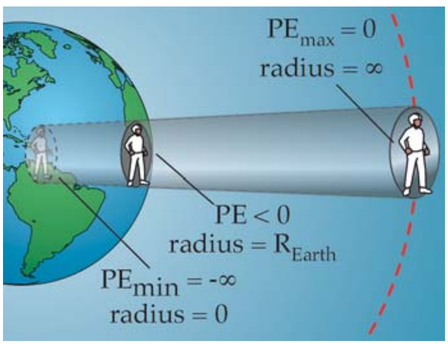


Figure 4.1.3-24. Potential Energy (PE). PE increases as we get farther from Earth's center by becoming less negative. It's as if we're climbing out of a deep well.

If you have a 1 kg mass suspended above your head, how do you realize the "potential" of its energy? You let go! Gravity will then cause the mass to accelerate downward, so when it hits the ground (and hopefully not your head or your foot, enroute), it's moving at considerable speed and thus has energy of a different kind—energy of motion, which we call kinetic energy. Similar to linear momentum, kinetic energy is solely a function of an object's mass and its velocity.

$$\boxed{KE = \frac{1}{2}mV^2} \qquad (4.1.3\text{-}14)$$

where

KE = kinetic energy ($kg\ km^2/s^2$)

m = mass (kg)

V = velocity (km/s)

As we said, total mechanical energy in a conservative field (such as a gravitational field) stays constant. But, a spacecraft in orbit may get close to Earth during part of its orbit and be far away in another part. So, how does it maintain a constant mechanical energy? It must trade the potential energy it loses as it moves closer, for kinetic energy (increased velocity). Then, as it goes farther away, it trades back—kinetic energy goes down, as the potential energy goes up.

The endless trade-off between PE and KE to make this happen goes on all around us—but we often don't notice it. We've all played on a simple playground swing like the one in Figure 4.1.3-25. As we swing back and forth, we constantly trade between KE and PE. At the bottom of the arc, we are moving the fastest, so our KE is at a maximum and PE is at a minimum. As we swing up, our speed diminishes until, at the top of the arc, we actually stop briefly. At this point, our KE is zero because we're not moving (velocity is zero), but our PE is at a maximum. The reverse happens as we swing back, this time turning our PE back into energy of motion. If it weren't for friction in the frame attachments and our own wind resistance, once we started on a swing, we'd swing forever even without "pumping."

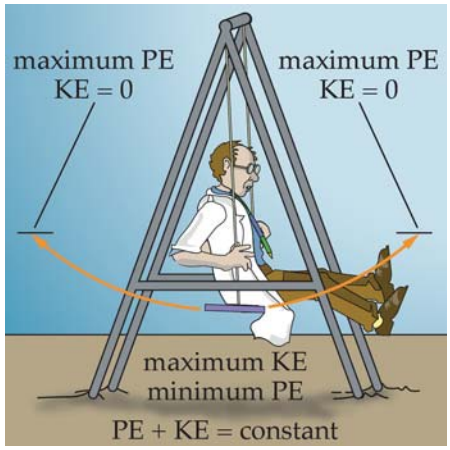


Figure 4.1.3-25. Mechanical Energy is Conserved. The total mechanical energy, the sum of kinetic and potential energy, is constant in a conservative field. We can show this with a simple swing. At the bottom of the arc, speed is greatest and height is lowest; hence, KE is at the maximum and PE is at a minimum. As the swing rises to the top of the arc, KE trades for PE until it stops momentarily at the top where PE is maximum and KE is zero.

We can now combine KE and PE to get a new expression for the total mechanical energy of our orbiting spacecraft

$$\boxed{E = \frac{1}{2}m\,V^2 - \frac{m\mu}{R}} \qquad (4.1.3\text{-}15)$$

where

E = total mechanical energy ($kg\ km^2/s^2$)

m = mass (kg)

V = velocity (km/s)

μ = gravitational parameter (km^3/s^2)

R = position (km)

Later we'll use this expression to develop some useful tools for analyzing orbital motion.

Section Review

Key Concepts

- A property is conserved if it stays constant in a system
- In the absence of outside forces, linear and angular momentum are conserved
- A conservative field, such as gravity, is one in which total mechanical energy is conserved
- Total mechanical energy, E, is the sum of potential and kinetic energies
  - Kinetic energy, KE, is energy of motion
  - Potential energy, PE, is energy of position

4.1.3.4 The Restricted Two-body Problem

In This Section You'll Learn to...

- Explain the approach used to develop the restricted two-body equation of motion, including coordinate systems and assumptions
- Explain how the solution to the two-body equation of motion dictates orbital geometry
- Define and use the terms that describe orbital geometry

Earlier, we outlined a general approach to analyzing the motion of an object called the MAP, shown again in Figure 4.1.3-26. There we described the motion of a baseball. Now we can use the first three steps of this same method to understand the motion of any object in orbit. A special application of the MAP is the *restricted two-body problem*. Why restricted? As we'll see later in this section, we must restrict our analysis with assumptions we need to make our lives easier. Why two bodies? That's one of the assumptions. Why a problem? Finding an equation to represent this motion has been a classic problem solved and refined by students and mathematicians since Isaac Newton. In this section, we'll rely on the work of the mathematicians who have come before us. So at the end of this section you'll say, "The motion of two bodies? Hey, no problem!"

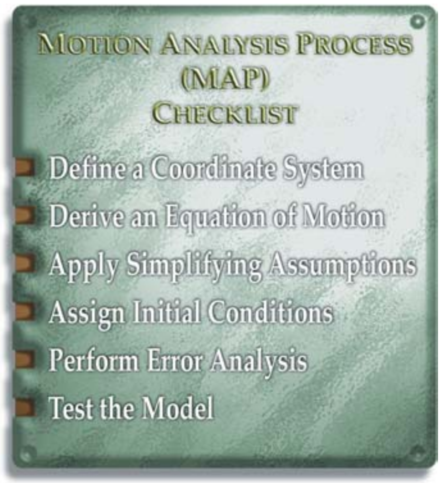


Figure 4.1.3-26. Motion Analysis Process (MAP) Checklist. Originally described in Section 4.1.3.1 (Figure 4.1.3-6), this process applies to balls in flight or spacecraft in orbit.

Coordinate Systems

To be valid, Newton's laws must be expressed in an inertial reference frame, meaning a frame that is not accelerating. To illustrate this, let's suppose we want to describe the flight of a baseball we toss and catch while we're driving in our car. We see the ball go up and down with respect to us. But that's not the whole story. Our car may be accelerating with respect to a police car behind us. Our car and the police car may be accelerating with respect to Earth's surface. And of course we must consider Earth's motion spinning on its axis, Earth's motion around the Sun, the Sun's motion in the Galaxy, the Galaxy's motion through the universe, and the expansion of the universe! These are all accelerating frames of reference for the ball's motion, which complicate our attempt to describe this motion using Newton's laws.

So we can see how this reference frame stuff can get complicated very quickly. Indeed, from astronomical observations, it looks like everything in the universe is accelerating. So how can we find any purely non-accelerating reference? We can't. To apply Newton's laws to our ball, we must select a reference frame that's close enough to, or "sufficiently," inertial for our problem.

Any reference frame is just a collection of unit vectors at right angles to each other that allows us to specify the magnitude and direction of other vectors, such as a spacecraft's position and velocity. This collection of unit vectors allows us to establish the components of vectors in 3-D space. By rigidly defining these unit vectors, we define a coordinate system.

To create a coordinate system we need to specify four pieces of information—an origin, a fundamental plane, a principal direction, and a third axis, as shown in Figure 4.1.3-27. The *origin* defines a physically identifiable starting point for the coordinate system. The other three parameters fix the orientation of the frame. The *fundamental plane* contains two axes of the system. Once we know the plane, we can establish the first axis by defining a unit vector that starts at the origin and is perpendicular to this plane. The unit vector in this direction at the origin is one axis. Next, we need a *principal direction* within the plane, which we define by pointing a unit vector toward some visible, distant object, such as a star. Now that we have two directions (the principal direction and an axis perpendicular to the fundamental plane), we can find the third axis using the right-hand rule.

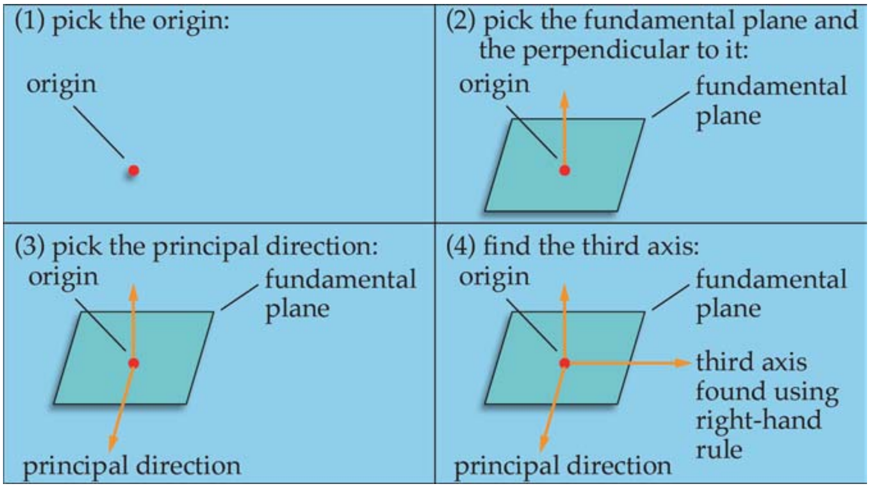


Figure 4.1.3-27. Defining a Coordinate System. We define coordinate systems by selecting a convenient (1) origin; (2) fundamental plane containing the origin and an axis perpendicular to the plane; (3) principal direction within the plane; and (4) third axis using the

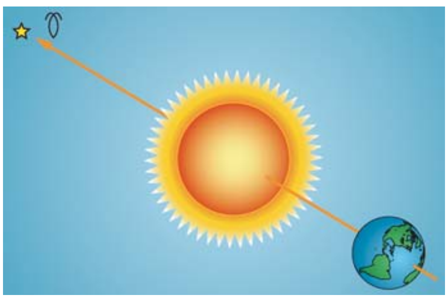


Figure 4.1.3-28. Vernal Equinox Direction. The vernal equinox direction is the principal direction for the geocentric-equatorial coordinate system. It's found by drawing a line from Earth through the Sun on the first day of Spring, usually March 21.

Remember—coordinate systems should make our lives easier. If we choose the correct coordinate system, developing the equations of motion can be simple. If we choose the wrong system, it can be nearly impossible.

For Earth-orbiting spacecraft, we'll choose a tried-and-true system that we know makes solving the equations of motion relatively easy. This *geocentric-equatorial coordinate system* has these characteristics

- Origin—Earth's center (hence the name *geo*centric)
- Fundamental plane—Earth's equator (hence geocentric-*equatorial*). Perpendicular to the plane—North Pole direction
- Principal direction—vernal equinox direction found by drawing a line from Earth to the Sun on the first day of Spring, as shown in Figure 4.1.3-28. While this direction may not seem "convenient" to you, it's

significant to astronomers who originally defined the system. Plus it beats any alternatives by a long way, mostly because they move.

- Third axis found using the right-hand rule

Figure 4.1.3-29 shows the entire coordinate system.

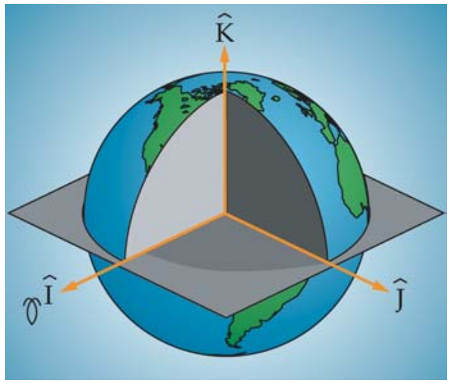


Figure 4.1.3-29. Geocentric-equatorial Coordinate System. We define this system by

- Origin—Earth's center
- Fundamental plane—equatorial plane
- Perpendicular to plane—North Pole
- Principal direction—vernal equinox (♈)

We use this coordinate system for analyzing the orbits of Earth-orbiting spacecraft.

Equation of Motion

Using the geocentric-equatorial coordinate system, we can safely apply Newton's Second Law to examine the external forces affecting the system, or in this case, a spacecraft. So let's place ourselves on an imaginary spaceship in orbit around Earth and see if we can list the forces on our ship.

- Earth's gravity (Newton wouldn't let us forget this one)
- Drag—if we're a little too close to the atmosphere
- Thrust—if we fire rockets
- 3rd body—gravity from the Sun, Moon, or planets
- Other—just in case we miss something

Summing all these forces, shown in Figure 4.1.3-30, we get with the following equation of motion

$$\sum \vec{F}_{external} = \vec{F}_{gravity} + \vec{F}_{drag} + \vec{F}_{thrust} + \vec{F}_{3rd\ body} + \vec{F}_{other} = m\vec{a} \quad (4.1.3\text{-}16)$$

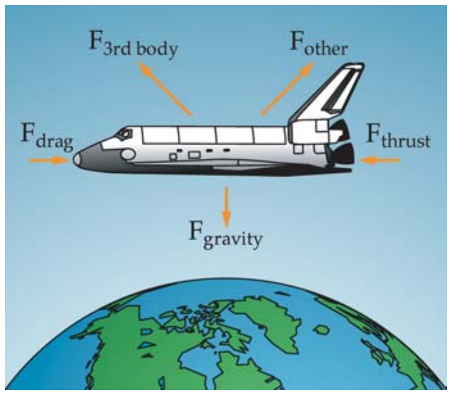


Figure 4.1.3-30. Possible Forces on a Spacecraft. We can brainstorm all the possible forces on a spacecraft to include Earth's gravity, drag, thrust, third-body gravity, and other forces.

If we substituted mathematical expressions for the various forces and tried to devise a solution to the equation, we would create a difficult problem—not to mention an enormous headache. So let's examine some reasonable assumptions we can make to simplify the problem.

Simplifying Assumptions

Luckily, we can assume some things about orbital motion that will simplify the problem, but they will "restrict" our solution to cases in which these assumptions apply. Fortunately, this includes most of the situations we'll use. Let's consider the forces on a spacecraft in orbit and assume

- The spacecraft travels high enough above Earth's atmosphere that the drag force is small, $\vec{F}_{drag} \cong 0$
- The spacecraft won't maneuver or change its path, so we ignore the thrust force, $\vec{F}_{thrust} \cong 0$
- We are considering the motion of the spacecraft close to Earth, so we ignore the gravitational attraction of the Sun, the Moon, or any other third body, $\vec{F}_{3rd\ body} \cong 0$. (That's why we call this the two-body problem.)
- Compared to Earth's gravity, other forces such as those due to solar radiation, electromagnetic fields, etc., are negligible, $\vec{F}_{other} \cong 0$

- Earth's mass is much, much larger than the mass of any spacecraft, $m_{Earth} >> m_{spacecraft}$
- Earth is spherically symmetrical with uniform density, so we treat it as a point mass. Thus, we mathematically describe Earth's gravity as acting from its center.
- The spacecraft's mass is constant, $\Delta m = 0$, so Equation (4.1.3-7), applies
- The geocentric-equatorial coordinate system is sufficiently inertial, so that Newton's laws apply

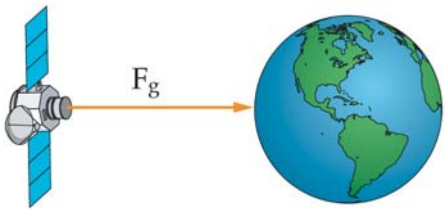


Figure 4.1.3-31. The Force of Gravity. In the restricted two-body problem, we reduce the forces acting on a spacecraft to a single force—Earth's gravity.

After all these assumptions, we're left with gravity as the only force, so our equation of motion becomes $\sum \vec{F}_{external} = \vec{F}_{gravity} = m\vec{a}$, as shown in Figure 4.1.3-31. Now we can apply Newton's Law of Universal Gravitation in vector form

$$\vec{F}_{gravity} = -\frac{\mu m}{R^2}\frac{\vec{R}}{R} \tag{4.1.3-17}$$

Substituting the force of gravity equation into the equation of motion, we get

$$\vec{F}_{gravity} = -\frac{\mu m}{R^2}\frac{\vec{R}}{R} = m\vec{a} = m\ddot{\vec{R}}$$

and dividing both sides by m, we arrive at the *restricted two-body equation of motion*

$$\boxed{\ddot{\vec{R}} + \frac{\mu}{R^2}\frac{\vec{R}}{R} = 0} \tag{4.1.3-18}$$

where

$\ddot{\vec{R}}$ = spacecraft's acceleration (km/s^2)

μ = gravitational parameter (km^3/s^2) = 3.986×10^5 km^3/s^2 for Earth

$\vec{R}$ = spacecraft's position vector (km)

R = magnitude of the spacecraft's position vector (km)

Note: We use the spacecraft's position vector, $\vec{R}$, divided by its magnitude, R, to denote the direction of the gravity force. Another way to show this direction is with a unit vector, $\hat{R}$. (See Appendix A for the definition of a unit vector.)

[*Note:* we use the engineering convention for the second derivative of $\vec{R}$ with respect to time, which is $\ddot{\vec{R}}$, better known as acceleration, $\vec{a}$.]

What can the two-body equation of motion tell us about the movement of a spacecraft around Earth? Unfortunately, in its present form—a second-order, non-linear, vector differential equation—it doesn't help us visualize anything about this movement. So what good is it? To understand the significance of the two-body equation of motion, we must first "solve" it, using a rather complex mathematical derivation. When the smoke clears, we're left with an expression for the magnitude of the position vector (not the velocity) of an object in space in terms of some odd, new variables.

$$R = \frac{k_1}{1 + k_2 \cos \nu}$$

where

R = magnitude of the spacecraft's position vector, $\vec{R}$

k_1 = constant that depends on μ, $\vec{R}$, and $\vec{V}$

k_2 = constant that depends on μ, $\vec{R}$, and $\vec{V}$

ν = (Greek letter "nu") polar angle measured from an orbit's principal axis to $\vec{R}$

This equation is the solution to the restricted, two-body equation of motion and describes the spacecraft's location, R, in terms of two constants and a polar angle, ν. You may recognize that this equation also represents a general relationship for any *circle*, *ellipse*, *parabola*, or *hyperbola*—commonly known as *conic sections*, shown in Figure 4.1.3-32. Now, here's the really significant part of all this—we just proved Kepler's Laws of Planetary Motion! Based on Brahe's data, Kepler showed that the planets' orbits were ellipses but couldn't say why. We've just shown why: any object moving in a gravitational field must follow one of the conic sections. In the case of planets or spacecraft in orbit, this path is an ellipse or a circle (which is just a special case of an ellipse).

Now that we know orbits must follow conic section paths, we can look at some ways to describe the size and shape of an orbit.

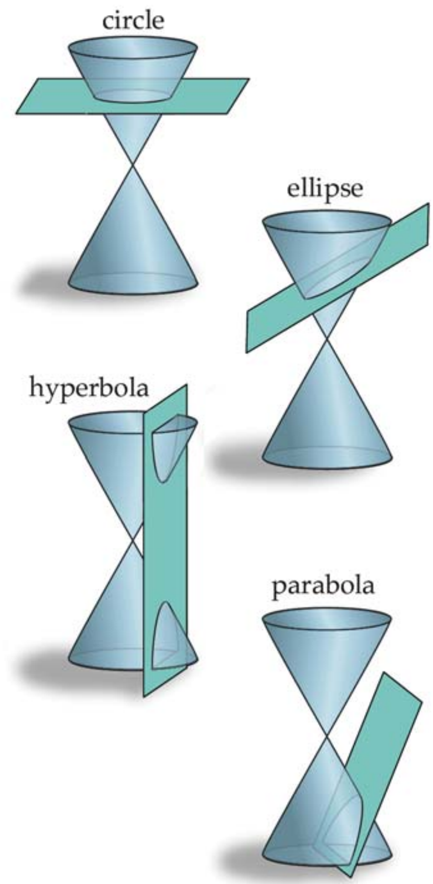
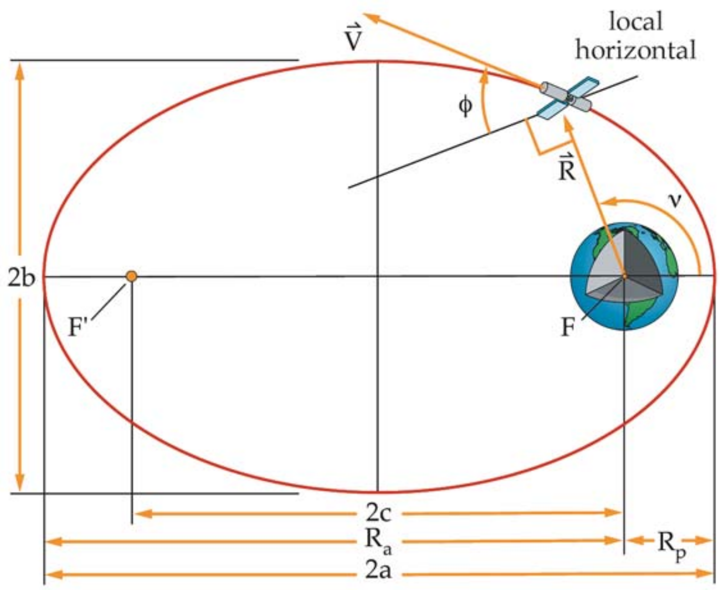


Figure 4.1.3-32. Conic Sections. The solution to the restricted, two-body equation of motion gives the polar equation for a conic section. Conic sections are found by slicing right cones at various angles.

Orbital Geometry

Because we're mainly interested in spacecraft orbits, which we know are elliptical, let's look closer at elliptical geometry. Using Figure 4.1.3-33 as a reference, let's define some important *geometrical parameters* for an ellipse.

- R is the radius from the focus of the ellipse (in this case, Earth's center) to the spacecraft
- F and F' are the *primary* (occupied) and *vacant* (unoccupied) *foci*. Earth's center is at the occupied focus.
- R_p is the *radius of periapsis* (radius of the closest approach of the spacecraft to the occupied focus); it's called the radius of *perigee* when the orbit is around Earth
- R_a is the *radius of apoapsis* (radius of the farthest approach of the spacecraft to the occupied focus); it's called the radius of *apogee* when the orbit is around Earth
- 2a is the major axis or the length of the ellipse. One-half of this is "a," or the *semimajor axis* (semi means one half).



$\vec{R}$ = spacecraft's position vector, measured from Earth's center

$\vec{V}$ = spacecraft's velocity vector

F and F' = primary and vacant foci of the ellipse

R_p = radius of perigee (closest approach)

R_a = radius of apogee (farthest approach)

2a = major axis

2b = minor axis

2c = distance between the foci

a = semimajor axis

b = semiminor axis

ν = true anomaly

ϕ = flight-path angle

Figure 4.1.3-33. Geometry of an Elliptical Orbit. With these parameters, we completely define the size and shape of the orbit.

$$a = \frac{R_a + R_p}{2} \tag{4.1.3-19}$$

- 2b is the minor axis or width of the ellipse. One-half of this is "b," or the *semiminor axis*.
- 2c is the distance between the foci, $R_a - R_p$
- ν is the *true anomaly* or polar angle measured from perigee to the spacecraft's position vector, $\vec{R}$, in the direction of the spacecraft's motion. It locates the spacecraft in the orbit. For example, if $\nu = 180°$ the spacecraft is 180° from perigee, putting it at apogee. The range for true anomaly is 0° to 360°.
- ϕ is the *flight-path angle*, measured from the local horizontal to the velocity vector, $\vec{V}$. At the spacecraft the local horizontal is a line perpendicular to the position vector, $\vec{R}$. When the spacecraft travels from perigee to apogee (outbound), its velocity vector is always above the local horizon (gaining altitude), so $\phi > 0°$. When it travels from apogee to perigee (inbound), its velocity vector is always below the local horizon (losing altitude), so $\phi < 0°$. At exactly perigee and apogee of an elliptical orbit, the velocity vector is parallel to the local horizon, so $\phi = 0$. The maximum value of the flight-path angle is 90°.
- e is the *eccentricity*, which is the ratio of the distance between the foci (2c) to the length of the ellipse (2a)

$$e = \frac{2c}{2a}$$

- Eccentricity defines the shape or type of conic section. Eccentricity is a medieval term representing a conic's degree of noncircularity (meaning "out of center"). Because circular motion was once considered perfect, any deviation was abnormal, or eccentric (maybe you know someone like that). Because the distance between the foci in an ellipse is always less than the length of the ellipse, its eccentricity is between 0 and 1. A circle has e = 0. A very long, narrow ellipse has e approaching 1. A parabola has e = 1 and a hyperbola has e > 1.

With all these geometrical parameters defined, let's look at our polar equation of a conic and substitute for the constants $k_1 = a\,(1 - e^2)$ and $k_2 = e$. Thus, we have

$$\boxed{R = \frac{a(1-e^2)}{1+e\cos\nu}} \tag{4.1.3-20}$$

where

R = magnitude of the spacecraft's position vector (km)
a = semimajor axis (km)
e = eccentricity (unitless)
ν = true anomaly (deg or rad)

To determine the distances at closest approach, R_p, and farthest approach, R_a, we can use this equation.

$$\text{At } \nu = 0°,\ R = R_p = \frac{a(1-e^2)}{(1+e\cos(0°))} = a\,(1-e)$$

$$\text{At } \nu = 180°,\ R = R_a = \frac{a(1-e^2)}{(1+e\cos(180°))} = a\,(1+e)$$

Looking at the geometry of an ellipse, we can see that the length of the ellipse, 2a, equals $(R_a + R_p)$, and the distance between the foci, 2c, is $(R_a - R_p)$. Now, if we want to compute the orbit's eccentricity based on the radii of perigee and apogee, we can use the second part of Equation (4.1.3-21).

$$e = \frac{2c}{2a} = \frac{R_a - R_p}{R_a + R_p} \tag{4.1.3-21}$$

Parameters for the ellipse also apply to circular orbits, parabolic trajectories, and hyperbolic trajectories. Figure 4.1.3-34 shows a circular orbit, where the radius from Earth's center is constant and equal to the semimajor axis. Therefore, this orbit has no apogee or perigee, and its eccentricity is zero. The flight path angle is always zero.

The parabola in Figure 4.1.3-35 represents a minimum escape trajectory or a path that just barely takes a spacecraft away from Earth, never to return. So there is no apogee and no empty focus. Thus, the semimajor axis and the distance between the foci are infinite. We say the eccentricity, e = 1. The true anomaly ranges from 0° to less than 180° on the outbound path. The flight path angle is greater than zero. Of course, if a spacecraft is inbound on a parabolic trajectory, its true anomaly is greater than 180°

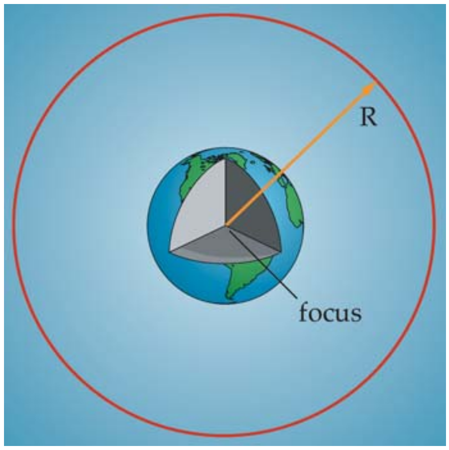


Figure 4.1.3-34. Circle. A circle is just a special case of an ellipse.

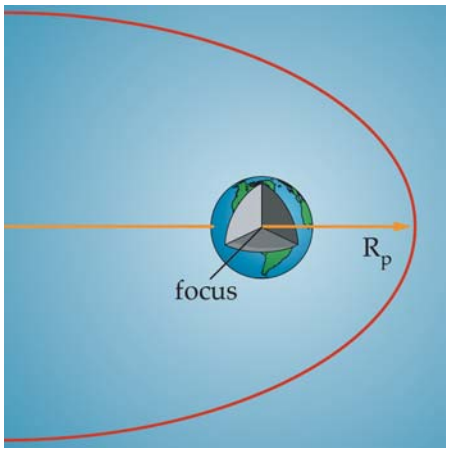


Figure 4.1.3-35. Parabola. A parabolic trajectory is a special case which leaves Earth

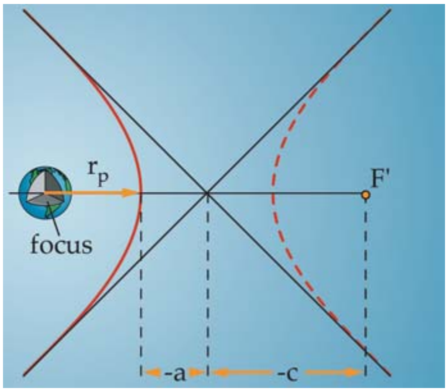


Figure 4.1.3-36. Hyperbola. We use a hyperbolic trajectory for interplanetary missions. Notice a real trajectory is around the occupied focus and an imaginary, mirror-image trajectory is around the vacant focus.

until it passes perigee, then it resets to 0° and grows to almost 180°. And its flight path angle is less than zero until it passes perigee.

The hyperbola in Figure 4.1.3-36 also represents an escape trajectory, so it also has no apogee. It's an unusual shape with a different sign convention. Because the length of the hyperbola (distance between the "ends") bends back on itself, or is measured outside the conic, we define this distance, 2a, as negative. The same convention also applies for the distance between the foci, 2c, so 2c is also negative. But the magnitude of 2c is always larger than the magnitude of 2a, so the eccentricity is greater than 1.0. The true anomaly ranges from 0° to less than 180° on the outbound path and greater than 180° to 0° on the inbound path. The flight path angle is greater than 0° on the outbound path and less than 0° on the inbound path. Table 4.1.3-1 summarizes these parameters.

Table 4.1.3-1. A Summary of Parameters for Conic Sections.

Conic Section	a = Semimajor Axis	c = One-half the Distance between Foci	e = Eccentricity
circle	$a > 0$	$c = 0$	$e = 0$
ellipse	$a > 0$	$0 < c < a$	$0 < e < 1$
parabola	$a = \infty$	$c = \infty$	$e = 1$
hyperbola	$a < 0$	$\|a\| < \|c\| > 0$	$e > 1$

Section Review

Key Concepts

- Combining Newton's Second Law and his Law of Universal Gravitation, we form the restricted two-body equation of motion
  - The coordinate system used to derive the two-body equation of motion is the geocentric-equatorial system
    - Origin—Earth's center
    - Fundamental plane—equatorial plane
    - Direction perpendicular to the plane—North Pole direction
    - Principal direction—vernal equinox direction
  - In deriving this equation, we assume
    - Drag force is negligible
    - Spacecraft is not thrusting
    - Gravitational pull of third bodies and all other forces are negligible
    - $m_{Earth} >> m_{spacecraft}$
    - Earth is spherically symmetrical and of uniform density and we can treat it mathematically as a point mass
    - Spacecraft mass is constant, so $\Delta m = 0$
    - The geocentric-equatorial coordinate system is sufficiently inertial for Newton's laws to apply
- Solving the restricted two-body equation of motion results in the polar equation for a conic section
- Figure 4.1.3-33 shows parameters for orbital geometry, and Table 4.1.3-1 summarizes parameters for conic sections

4.1.3.5 Constants of Orbital Motion

In This Section You'll Learn to...

- Define the two constants of orbital motion—specific mechanical energy and specific angular momentum
- Apply specific mechanical energy to determine orbital velocity and period
- Apply the concept of conservation of specific angular momentum to show an orbital plane remains fixed in space

By now you're probably convinced that, with all these flight-path angles, true anomalies, and ellipses flying around, there is nothing consistent about orbits. Well, take heart because we do have constants in astrodynamics. We saw in our discussion of motion in a conservative field that mechanical energy and momentum are conserved. Because orbital motion occurs in a conservative gravitational field, spacecraft conserve mechanical energy and angular momentum. So, now let's see how these principles provide valuable tools for studying orbital motion.

Specific Mechanical Energy

In an earlier section, we referred to equations of motion being like crystal balls, in that they allow us to gaze into the future to predict where an object will be. Mechanical energy provides us with one such crystal ball. Recall in defining mechanical energy, we add potential energy, PE, to kinetic energy, KE. Together, they form a relationship between a spacecraft's mass, m, its position, R, its velocity, V, and the local gravitational parameter, μ ($3.986 \times 10^5\ \mathrm{km^3/s^2}$ for Earth).

$$E = \frac{1}{2}mV^2 - \frac{\mu m}{R} \qquad (4.1.3\text{-}22)$$

To generalize this equation, so we don't have to worry about mass, let's divide both sides of the equation by m. Doing so defines a new flavor of mechanical energy called *specific mechanical energy, ε*, which doesn't depend on mass. Thus, we can talk about the energy in a particular orbit, whether the orbiting object is a golf ball or the International Space Station. Specific mechanical energy, ε, is simply the total mechanical energy divided by a spacecraft's mass

$$\varepsilon \equiv \frac{E}{m} \qquad (4.1.3\text{-}23)$$

where $\equiv$ means "defined as," or

$$\boxed{\varepsilon = \frac{V^2}{2} - \frac{\mu}{R}} \qquad (4.1.3\text{-}24)$$

where

ε = spacecraft's specific mechanical energy (km^2/s^2)

V = spacecraft's velocity (km/s)

μ = gravitational parameter (km^3/s^2) = 3.986×10^5 km^3/s^2 for Earth

R = spacecraft's distance from Earth's center (km)

Because the specific mechanical energy is conserved, it must be the same at any point along an orbit! As a spacecraft approaches apogee, it is gaining altitude, meaning its R, or distance from Earth's center, increases. This increase in R means it gains potential energy—which actually means the potential energy (PE) gets less negative (because of the way we define it). At the same time, the spacecraft's speed is decreasing and hence it is losing kinetic energy (KE). When it reaches the highest point, its PE is at a maximum. However, because its speed is the slowest at apogee, KE is at a minimum. But the sum of PE and KE—specific mechanical energy—remains constant.

As the spacecraft passes apogee and starts toward perigee, it begins to trade its PE for KE. So, its speed steadily increases until it reaches perigee, where its speed is fastest and its KE is maximum. Again, the sum of potential and kinetic energy—specific mechanical energy—remains constant. Figure 4.1.3-37 illustrates these relationships.

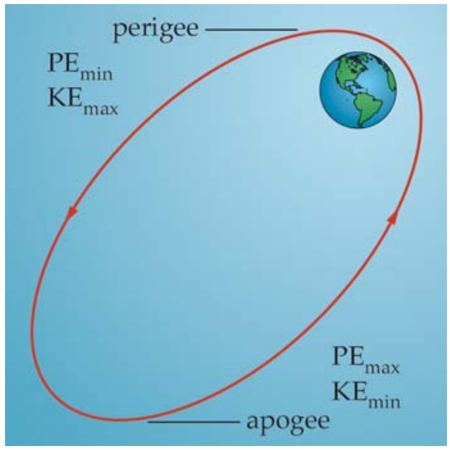


Figure 4.1.3-37. Trading Energy in an Orbit. An orbit is just like a swing. PE and KE trade-off throughout the orbit, so their sum is constant.

The fact that the specific mechanical energy is constant gives us a tremendously powerful tool for analyzing orbits. Look again at the relationship for specific mechanical energy, ε. Notice that ε depends only on position, R, velocity, V, and the local gravitational parameter, μ. This means if we know a spacecraft's position and velocity at any point along its orbit, we know its specific mechanical energy for every point on its orbit.

Another important concept to glean out of the constancy of orbital energy is the relationship between R and V. Assume we know the energy for an orbit. Then, at any given position, R, on that orbit, there is only one possible velocity, V! Thus, if we know the orbital energy and R, we can easily find the velocity at that point. Simply rearranging the relationship for energy gives us an extremely useful expression for velocity.

$$\boxed{V = \sqrt{2\left(\frac{\mu}{R} + \varepsilon\right)}} \qquad (4.1.3\text{-}25)$$

where

V = spacecraft's velocity (km/s)

μ = gravitational parameter (km^3/s^2) = 3.986×10^5 km^3/s^2 for Earth

R = spacecraft's distance from Earth's center (km)

ε = spacecraft's specific mechanical energy (km^2/s^2)

We often use this equation to determine velocities while analyzing orbits. For example, during space missions we often have to move spacecraft from one orbit to another. We can use this relationship to determine how much we must change the velocity to "drive" over to the new orbit.

Recall from our discussion of conic-section geometry, one parameter represents a spacecraft's mean, or average, distance from the primary focus. This parameter is the semimajor axis, a. We can develop a new relationship for specific mechanical energy which depends only on a and μ.

$$\boxed{\varepsilon = -\frac{\mu}{2a}} \qquad (4.1.3\text{-}26)$$

where

ε = spacecraft's specific mechanical energy (km^2/s^2)

μ = gravitational parameter (km^3/s^2) = 3.986×10^5 km^3/s^2 for Earth

a = semimajor axis (km)

This means simply knowing the semimajor axis of a spacecraft's orbit tells us its specific mechanical energy. We can also learn the type of trajectory from the sign of the specific mechanical energy, ε. For a circular or elliptical orbit, ε is *negative* (because a is positive). For a parabola, $\varepsilon = 0$ (because $a = \infty$). For a hyperbola, ε is *positive* (because a is negative). These are important points to keep in mind as we work orbital problems. If the sign for ε is wrong, the answer probably will be wrong.

Another benefit to knowing a value for energy is that we can determine orbital period. The *orbital period, P*, is the time it takes for a spacecraft to revolve once around its orbit. From Kepler's Third Law of Planetary Motion, P^2 is proportional to a^3, where "a," is the semimajor axis. Using this relationship, we can derive an expression for the orbital period

$$\boxed{P = 2\pi\sqrt{\frac{a^3}{\mu}}} \qquad (4.1.3\text{-}27)$$

where

P = period (seconds)

π = 3.14159. . .(unitless)

a = semimajor axis (km)

μ = gravitational parameter (km^3/s^2) = 3.986×10^5 km^3/s^2 for Earth

Notice that period only has meaning for "closed" conics (circles or ellipses). Period is infinite for a parabola, whose semimajor axis is infinite, and it's an imaginary number for a hyperbola, whose semimajor axis is negative.

Specific mechanical energy, ε, is a very valuable constant of spacecraft motion. With a single observation of position and velocity, we learn much about a spacecraft's orbit.

But ε gives us only part of the story. It tells us the orbit's *size* but doesn't tell us anything about where the orbit is in space. For insight into that important bit of information we need to look at the angular momentum.

Specific Angular Momentum

Recall from our discussion in Section 4.1.3.2 that we can find angular momentum from Equation (4.1.3-3).

$$\vec{H} = \vec{R} \times m\vec{V}$$

Once again, to uncomplicate our life, we divide both sides of the equation by the mass, m, of the object we're investigating. Doing this, we define the *specific angular momentum, $\vec{h}$*, as

$$\vec{h} \equiv \frac{\vec{H}}{m}$$

where

$\equiv$ means "defined as," or

$$\boxed{\vec{h} = \vec{R} \times \vec{V}} \qquad (4.1.3\text{-}28)$$

where

$\vec{h}$ = spacecraft's specific angular momentum vector (km^2/s)

$\vec{R}$ = spacecraft's position vector (km)

$\vec{V}$ = spacecraft's velocity vector (km/s)

Notice that specific angular momentum is the result of the cross product between two vectors: position and velocity. Recall from geometry that any two lines define a plane. So in this case, $\vec{R}$ and $\vec{V}$ are two lines (vectors having magnitude and direction) that define a plane. We call this plane containing $\vec{R}$ and $\vec{V}$, the *orbital plane*. Because the cross product of any two vectors results in a third vector that is perpendicular to the first two, the angular momentum vector $\vec{h}$ must be perpendicular to $\vec{R}$ and $\vec{V}$. Figure 4.1.3-38 shows $\vec{R}$, $\vec{V}$, and $\vec{h}$.

Here's where we need to apply a little deductive reasoning and consider the logical consequence of the facts we know to this point. First of all, as we saw in Section 4.1.3.3, angular momentum and, hence, specific angular momentum are constant in magnitude and direction. Second, $\vec{R}$ and $\vec{V}$ define the orbital plane. Next, $\vec{h}$ is perpendicular to the orbital plane. Therefore, if $\vec{h}$ is always perpendicular to the orbital plane, and $\vec{h}$ is constant, the orbital plane must also be constant. This means that in our restricted, two-body problem the orbital plane is forever frozen in inertial space! However, in reality, slight disturbances cause the orbital plane to change gradually over time.

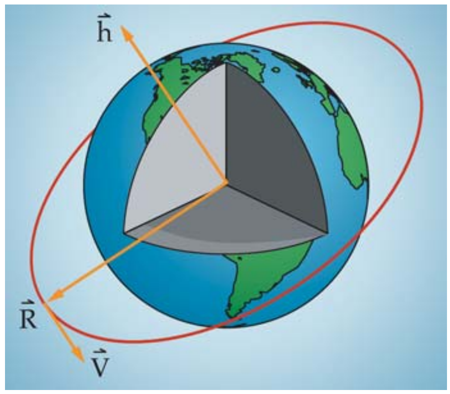


Figure 4.1.3-38. Specific Angular Momentum. The specific angular momentum vector, $\vec{h}$, is perpendicular to the orbital plane defined by $\vec{R}$ and $\vec{V}$.

Section Review

Key Concepts

- In the absence of any force other than gravity, two quantities remain constant for an orbit
  - Specific mechanical energy, ε
  - Specific angular momentum, $\vec{h}$
- Specific mechanical energy, ε, is defined as $\varepsilon \equiv E/m$
  - $\varepsilon < 0$ for circular and elliptical orbits
  - $\varepsilon = 0$ for parabolic trajectories
  - $\varepsilon > 0$ for hyperbolic trajectories
- Specific angular momentum, $\vec{h}$ is defined as $\vec{h} \equiv \vec{H}/m$
  - It is constant for an orbit
  - Because $\vec{h}$ is constant, orbital planes are fixed in space (neglecting orbital perturbations)

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